THE CHINESE UNIVERSITY OF HONG KONG
MATH 2230 (Second Term, 2023-24)
Complex Variables with Applications
Midterm 2
Time: 7-8 p.m, 2nd April 2024

Answer all FOUR questions.

1. (20 pts) Evaluate the line integral over $C : |z| = 2$

   (a) $\int_C \frac{e^{3z}}{(z-1)^3}\, dz$

   (b) $\int_C \frac{e^{3z}}{3z^2-10z+3}\, dz$

2. (30 pts) For each of the statement below, if the answer is yes, write down an explicit example and the function it converges to; if the answer is no, briefly explain the reason.

   (a) Can a series $\sum_{n=0}^{\infty} a_n(z-2)^n$ converge at $z = 0$ and diverge at $z = 3$?

   (b) Can a series $\sum_{n=0}^{\infty} a_n(z-2)^n$ diverge at $z = 0$ and converge at $z = 3$?

3. (30 pts) Let $f(z) = \frac{1}{(z-1)(z-3)}$. Find $0 < r_1 < r_2$ such that $f(z)$ is analytic on the three annular regions $A_1 : \{|z| < r_1\}$, $A_2 : \{r_1 < |z| < r_2\}$ and $A_3 : \{r_2 < |z|\}$, and find its Laurent series on each of the above annuli.

4. (20 pts) Let $f(z) = e^{\cos(z)} z^{2024}$. Let D be the disk $|z-4| \leq 3$. Show that $f(z)$ attains both its maximum and minimum modulus in D on the circle $|z-4| = 3$.

Midterm 2

1. (a) $\int_C \frac{e^{3z}}{(z-1)^3}\, dz = \frac{2\pi i}{2!} (e^{3z})'' \Big|_{z=1} = 9e^3\pi i$

(b) $\int_C \frac{e^{3z}}{3z^2-10z+3}\, dz = \int_C \frac{\frac{1}{3} e^{3z}/(z-3)}{z-\frac{1}{3}}\, dz$

$= 2\pi i \cdot \frac{e^{3z}}{3(z-3)} \Big|_{z=\frac{1}{3}} = -\frac{1}{4} e\pi i$

(Cauchy's Integral Formula)

2. (a) No. The series is centred at $z=2$.

Since $|0-2| = 2 < 1 = |3-2|$, it is impossible.

(b) Yes.

A simple example:

Let $a_n = 2^{-n}$

At $z=0$:

$\sum_{n=0}^{\infty} a_n (z-2)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges.

At $z=3$:

$\sum_{n=0}^{\infty} a_n (z-2)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$

It converges to $\frac{1}{1-\frac{1}{2}(z-2)} = 2/(4-z)$

(Radius of convergence)

3. $r_1 = 1 \qquad r_2 = 3$ (6 pts)

For $z \in A_1$,
$$f(z) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{3-z}\right)$$
$$= \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}\right)$$
$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(1-\left(\frac{1}{3}\right)^{n+1}\right) z^n \qquad \text{(8 pts)}$$

For $z \in A_2$,
$$f(z) = \frac{1}{2}\left(-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}\right)$$
$$= \frac{1}{2}\left(-\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right)$$
$$= -\frac{1}{2}\left(\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}\right) \qquad \text{(8 pts)}$$

For $z \in A_3$,
$$f(z) = \frac{1}{2z}\left(\frac{1}{1-\frac{3}{z}} - \frac{1}{1-\frac{1}{z}}\right)$$
$$= \frac{1}{2z}\left(\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \left(\frac{1}{z}\right)^n\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(3^n - 1\right)\cdot\left(\frac{1}{z}\right)^{n+1} \qquad \text{(8 pts)}$$

4. Maximum & Minimum Modulus Principle:

f is non-constant, holomorphic and non-zero on D.